Unit 3: Design of Digital Filters

Structures for FIR and IIR Systems:

Structure for FIR Systems:

In general a FIR system is described by the difference equation

\[ y(n) = \sum_{k=0}^{M-1} b_k x(n-k) \]

Or equivalently, by the system function

\[ H(z) = \sum_{k=0}^{M-1} b_k z^{-k} \]

1. Direct-Form Structure:

The direct-form realization follows the convolution summation

\[ y(n) = \sum_{k=0}^{M-1} h(k)x(n - k) \]

We observe that this structure requires M-1 memory locations for storing the M-1 previous inputs, and has a complexity of M multiplications and M-1 additions per output point. Since the output consists of a weighted linear combination of M-1 past values of the input and the weighted current value of the input, the structure in above figure, resembles a tapped delay line or a transversal system consequently, the direct-form realization is often called a transversal or tapped-delay-line filter.
2. **Cascade-Form Structures:**

The cascade realization follows naturally from the system function given by

\[ H(z) = \sum_{k=0}^{M-1} b_k z^{-k} \]

It is simple matter to factor \( H(z) \) into second order FIR system so that

\[ H(z) = \prod_{k=1}^{K} H_k(z) \]

Where \( H_k(z) = b_{k0} + b_{k1} z^{-1} + b_{k2} z^{-2}, k = 1, 2, 3, \ldots \)

And \( K \) is the integer part of \( (M + 1) / 2 \). The filter parameter \( b_0 \) may be equally distributed among the \( K \) filter sections, such that \( b_0 = b_{10} b_{20} \cdots b_{K0} \) or it may be assigned to a single filter section. The zeros of \( H(z) \) are grouped in pairs to produce the second-order FIR systems. It is always desirable to form pairs of complex-conjugate roots so that the coefficients \( \{ b_{ki} \} \) are real valued. On the other hand, real-valued roots can be paired in any arbitrary manner. The cascade-form realization along with the basic second-order section is shown below.

Cascade Realisation of a FIR system

![Cascade Realisation of a FIR system](image)
Design of Digital Filters:

Causality and Its Implications:

Let us consider the issue of causality in more detail by examining the impulse response $h(n)$ of an ideal low pass filter with frequency response characteristic $H(w)$:

$$H(w) = \begin{cases} 1 & |\omega| \leq \omega_c \\ 0 & \omega_c < |\omega| < \pi \end{cases}$$

The impulse response of the filter is

$$h(n) = \begin{cases} \frac{\omega_c}{\pi}, n = 0 \\ \frac{\omega_c}{\pi} \sin \frac{\omega_c n}{\pi}, n \neq 0 \end{cases}$$

A plot of $h(n)$ for $\omega_c = \pi/4$ is illustrated in the above figure. It is clear that the ideal low pass filter is noncausal and hence it cannot be realized in practice.

One possible solution is to introduce a large delay $n_0$ in $h(n)$ and arbitrarily to set $h(n)=0$ for $n < n_0$. However, the resulting system no longer has an ideal frequency response characteristic. Indeed, if we set $h(n) = 0$ for $n < n_0$, the Fourier series expansion of $H(w)$ results in the Gibbs phenomenon.
Paley-Wiener Theorem:

If \( h(n) \) has finite energy and \( h(n) = 0 \) for \( n < 0 \), then

\[
\int_{-\pi}^{\pi} \ln |H(\omega)| d\omega < \infty
\]

Conversely, if \( |H(\omega)| \) is square integrable and if the integral in the above equation is finite, then we can associate with \( H(\omega) \) a phase response \( \Theta(\omega) \), so that the resulting filter with frequency response \( H(\omega) = |H(\omega)| e^{j\Theta(\omega)} \) is causal.

One important conclusion that we draw from the Paley-Wiener theorem is that the magnitude function \( |H(\omega)| \) can be zero at some frequencies, but it can’t be zero over any finite band of frequencies, since the integral then becomes infinite. Consequently any ideal filter is noncausal.

Apparently causality imposes some tight constraints on a linear time invariant system. In addition to the Paley-Wiener condition causality also implies a strong relation between \( H_R(\omega) \) and \( H_I(\omega) \), the real and imaginary components of the frequency response \( H(\omega) \). To illustrate this dependence we decompose \( h(n) \). That is even and an odd sequence, that is

\[
H(n) = h_e(n) + h_o(n)
\]

Where \( h_e(n) = \frac{1}{2} [h(n) + h(-n)] \) and \( h_o(n) = \frac{1}{2} [h(n) - h(-n)] \)

Now, if \( h(n) \) is causal, it is possible to recover \( h(n) \) from its even part \( h_e(n) \) for \( 0 \leq n \leq \infty \) or from its odd component \( h_o(n) \) for \( 1 \leq n \leq \infty \).

Indeed, it can be easily seen that

\[
h(n) = 2h_e(n)u(n) - h_e(0)\delta(n) \quad n \geq 0
\]

and

\[
h(n) = 2h_o(n)u(n) - h_o(0)\delta(n) \quad n \geq 1
\]

Since \( h_o(n) = 0 \) for \( n = 0 \), we cannot recover \( h(0) \) from \( h_o(n) \) and hence we also must know \( h(0) \). In any case, it is apparent that \( h_o(n) = h_e(n) \) for \( n > 1 \), so there is a strong relationship between \( h_0(n) \) and \( h_e(n) \).
If \( h(n) \) is absolutely summable (i.e., BIBO stable), the frequency response \( H(\omega) \) exists, and

\[
H(\omega) = H_R(\omega) + jH_I(\omega)
\]

In addition, if \( h(n) \) is real valued and causal, the symmetry properties of the Fourier transform imply that

\[
H_e(n) \leftrightarrow H_R(\omega)
\]

\[
H_o(n) \leftrightarrow H_I(\omega)
\]

Since \( h(n) \) is completely specified by \( h_e(n) \), it follows that \( H(\omega) \) is completely determined if we know \( H_R(\omega) \). Alternatively, \( H(\omega) \) is completely determined from \( H_I(\omega) \) and \( h(0) \). In short, \( H_R(\omega) \) and \( H_I(\omega) \) are independent and cannot be specified independently if the system is causal. Equivalently, the magnitude and phase responses of a causal filter are interdependent and hence cannot be specified independently.

**Design of Linear Phase FIR filters using different windows:**

In many cases a linear phase characteristics is required through the passband of the filter. It can be shown that causal IIR filter cannot produce a linear phase characteristics and only special forms of causal FIR filters can give linear phase. If \( \{h[n]\} \) represents the impulse response of a discrete time linear system a necessary and sufficient condition for linear phase is that \( \{h[n]\} \) have finite duration \( N \), that it be symmetric about its midpoint, i.e.

\[
h[n] = h[N - 1 - n], \quad n = 0, 1, 2, \ldots (N - 1)
\]

\[
H(e^{j\omega}) = \sum_{n=0}^{N-1} h[n] e^{-j\omega n}
\]

\[
= \sum_{n=0}^{N/2-1} h[n] e^{-j\omega n} + \sum_{n=N/2}^{N-1} h[n] e^{-j\omega n}
\]

\[
= \sum_{n=0}^{N/2-1} h[n] e^{-j\omega n} + \sum_{m=0}^{N/2-1} h[m] e^{-j\omega (N - 1 - m)}
\]

For \( N \) even, we get

\[
H(e^{j\omega}) = e^{-j\omega (N-1)/2} \sum_{n=0}^{N/2-1} 2h[N] \cos(\omega (n - (N - 1)/2))
\]

For \( N \) odd

\[
H(e^{j\omega}) = e^{-j\omega (N-1)/2} \left\{ h[N/2] + \sum_{n=0}^{N-1} 2h[n] \cos(\omega (n - \frac{N-1}{2})) \right\}
\]
For N even we get a non-integer delay, which will cause the value of the sequence to change.

One approach to design FIR filters linear phase is to use windows. The easiest way to obtain FIR filter is to simply truncate the impulse response of an IIR filter. If \{h_d[n]\} is the impulse response of the designed FIR filter then the fir filter with impulse response \{h[n]\} can be obtained as follows.

\[
H[n] = \begin{cases} 
  h_d[n], & N_1 \leq n \leq N_2 \\
  0, & otherwise 
\end{cases}
\]

This can be thought of as being formed by a product of \{h_d[n]\} and a window function \{w[n]\}

\{h[n]\} = \{h_d[n]\} \{w[n]\}

where \{w[n]\} is the window function.

Using modulation property of fourier transform

\[H(e^{j\omega}) = \frac{1}{2\pi} \left[ H_d(e^{j\omega}) \star w(e^{j\omega}) \right]\]

In general for smaller N values spreading of main lobe more, and for larger N narrower thr main lobe and \(|H(e^{j\omega})|\) comes closer to \(|H_d(e^{j\omega})|\). Much work has been done on adjusting \{w[n]\} to satisfy certain main lobe and side lobe requirements. Some of the commonly used windows are given below-

(a) Rectangular Window

\[W_{R}(n) = \begin{cases} 
  1, & 0 \leq n \leq N - 1 \\
  0, & otherwise 
\end{cases}\]

(b) Bartlett (Triangular)

\[W_{B}(n) = \begin{cases} 
  \frac{2n}{N-1}, & 0 \leq n \leq (N - 1)/2 \\
  2 - \frac{2n}{N-1}, & (N - 1)/2 \leq n \leq N - 1 \\
  0, & elsewhere 
\end{cases}\]
(c) Hanning Window

\[ W_{\text{Han}}(n) = \begin{cases} \frac{1 - \cos[\pi m/(N - 1)]}{2} & 0 \leq n \leq N - 1 \\ 0, & \text{otherwise} \end{cases} \]

(d) Blackman Window

\[ W_{\text{Bl}}(n) = \begin{cases} 0.42 - 0.5 \cos\left\{ \frac{2 \pi n}{N - 1} \right\} + 0.08 \cos\left\{ \frac{4 \pi n}{N - 1} \right\} & 0 \leq n \leq N - 1 \\ 0, & \text{otherwise} \end{cases} \]

(e) Kaiser Window

\[ W_{K}(n) = \begin{cases} \frac{I_{0}(\omega_n \left[ \left( \frac{N - 1}{2} \right)^2 - \left( \frac{n - N - 1}{2} \right)^2 \right]^{1/2}}{I_{0}(\omega_{0} \left( \frac{N - 1}{2} \right))} & 0 \leq n \leq N - 1 \\ 0, & \text{otherwise} \end{cases} \]

Where \( I_{0}(x) \) is the modified Zero Order Bessel Function of the first kind.

The Transition width and the minimum stopped attenuation for different windows are listed below-

<table>
<thead>
<tr>
<th>Window</th>
<th>Transition Width</th>
<th>Minimum stopband attenuation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rectangular</td>
<td>( 4\pi/N )</td>
<td>-21dB</td>
</tr>
<tr>
<td>Bartlett</td>
<td>( 8\pi/N )</td>
<td>-25dB</td>
</tr>
<tr>
<td>Hanning</td>
<td>( 8\pi/N )</td>
<td>-44dB</td>
</tr>
<tr>
<td>Hamming</td>
<td>( 8\pi/N )</td>
<td>-53dB</td>
</tr>
<tr>
<td>Blackman</td>
<td>( 12\pi/N )</td>
<td>-74dB</td>
</tr>
<tr>
<td>Kaiser</td>
<td>variable</td>
<td>variable</td>
</tr>
</tbody>
</table>

We first choose a window that satisfies the minimum attenuation and the bandwidth that allows us to choose the appropriate value of \( N \). Actual frequency response characteristics are then calculated and we check the requirements are met or not.

**Design of IIR Filters:**

There are two methods for design the IIR filter.

1. Impulse Invariant Method
2. Bilinear Transformation Method
1. **Filter design by impulse invariance:**

Here the impulse response $h[n]$ of the desire discrete time system is proportional to equally spaces samples of the continuous time filter i.e,

$$H[n] = T_d h_a(nT_d)$$

Where $T_d$ represents a sample interval. Since the specification of the filter are given in discrete time domain it turns out that $T_d$ has no role to play in design of the filter. From the sampling theorem the frequency response of the discrete time filter is given by

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} H_a(j \frac{\omega}{T_d} + j \frac{2\pi k}{T_d})$$

Since any practical continuous time filter is not strictly band limited there is some aliasing. However if the continuous time filter approaches zero at high frequency the aliasing may be negligible. Then the frequency response of the discrete time filter is

$$H(e^{j\omega}) \approx \sum_{k=-\infty}^{\infty} H_a(j \frac{\omega}{T_d}), \quad |\omega| \leq \pi$$

We first convert digital filter specifications to continuous time filter specifications. Neglecting aliasing we get $H_a(j\Omega)$ specification by applying the relation $\Omega = \omega / T_d$. Where $H_a(j\Omega)$ is transferred to the designed filter $H(z)$.

Let us assume that the poles of the continuous time filter are simple, then

$$H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s - s_k}$$

The corresponding Impulse response is $h_a(t) = \sum_{k=1}^{N} A_k e^{s_k t}, \quad t \geq 0$

$$0, \quad t < 0$$

Then $h[n] = T_d h_a(nT_d) = \sum_{k=1}^{N} T_d A_k e^{s_k nT_d} u[n]$  

The system function function for this is $H(z) = \sum_{k=1}^{N} \frac{T_d A_k}{1 - e^{s_k T_d} z^{-1}}$

We see that a pole at $s = s_k$ in the $s$-plane is transferred to a pole at $z = e^{s_k T_d}$ in the $z$-plane. If the continuous time filter is stable i.e $\text{Re}\{s_k\} < 0$, then the magnitude of $e^{s_k T_d}$ will be less than 1. So the pole will be inside the unit circle. Thus the causal discrete filter is stable. The mapping of zero is not so straightforward.
Bilinear Transformation:

This technique avoids the problem of aliasing by mapping \( j\Omega \) axis in the s-plane to one revolution of unit circle in the z-plane. If \( H_a(s) \) is the continuous time transfer function the discrete time transfer function is detainted by replacing \( s \) with

\[
S = \frac{2}{T_d} \left( \frac{1-z^{-1}}{1+z^{-1}} \right)
\]

From which we get

\[
z = \frac{1+(T_d/2)s}{1-(T_d/2)s}
\]

Substituting \( s = \sigma + j\Omega \), we get

\[
z = \frac{1+\frac{T_d}{2}\sigma + \frac{j\Omega T_d}{2}}{1-\frac{T_d}{2}-j\frac{\Omega T_d}{2}}
\]

If \( \sigma < 0 \), it is then magnitude of the real part in the denominator is more than that of the numerator and so \( |z| < 1 \). Similarly if \( \sigma > 0 \) then \( |z| > 1 \) for all \( \Omega \). Thus pole in the left half of the s-plane will get mapped to the poles inside the unit circle in z-plane. If \( \sigma = 0 \) then

\[
z = \frac{1 + \frac{j\Omega T_d}{2}}{1 - j\frac{\Omega T_d}{2}}
\]

so \( |z| = 1 \), writing \( z = e^{j\omega} \) we get

\[
e^{j\omega} = \frac{1 + \frac{j\Omega T_d}{2}}{1 - j\frac{\Omega T_d}{2}}
\]

Rearranging we get

\[
j \frac{\Omega T_d}{2} = \frac{e^{j\omega/2} - 1}{e^{j\omega/2} - 1} = \frac{e^{j\omega/2} (e^{j\omega/2} - e^{-j\omega/2})}{e^{j\omega/2} (e^{j\omega/2} + e^{-j\omega/2})} = \frac{j}{\sin\omega/2} \frac{\sin\omega/2}{\cos\omega/2}
\]

Or \( \Omega = \frac{2}{T_d} \tan\frac{\omega}{2} \) or \( \omega = 2 \tan^{-1}\frac{\Omega T_d}{2} \).